

MAS231 GEOMETRY II (B.Sc. examination by course units)

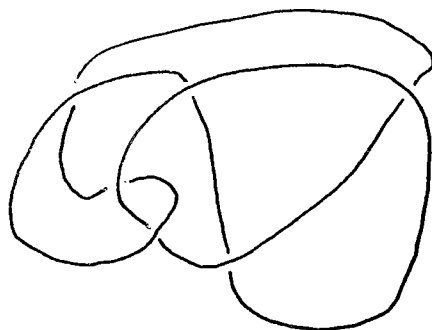
Duration: 2 hours

Date and time: To be determined

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best FOUR questions will be counted.

Question 1. (25 marks)

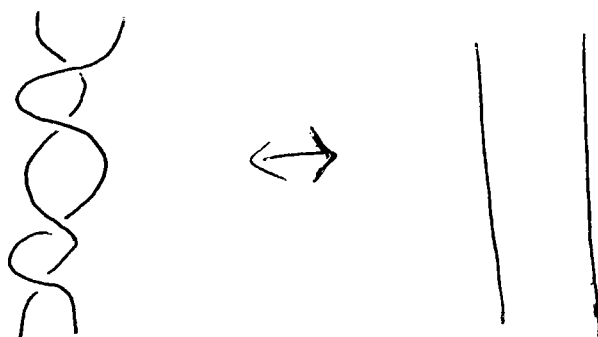
(a) (10 marks) Determine whether or not the knot



is tricolourable.

(b) (8 marks) Prove that Reidemeister move 2 is invariant for tricolourability.

(c) (7 marks) Suppose that, in addition to the three Reidemeister moves, we introduce the following “Queen Mary” move:



Determine whether or not the Queen Mary move is invariant for writhe.

Question 2. (25 marks)

- (a) (3 marks) State which of the three Reidemeister moves are invariant for the Kauffman Bracket.
- (b) (10 marks) Determine the Kauffman Bracket of the knot



You may use the fact that

$$B \left(\text{trefoil} \right) = x^{-7} - x^{-3} - x^5$$

- (c) (8 marks) Use your answer in part (b) to determine the Jones polynomial $J(t)$ of the knot in part (b).
- (d) (4 marks) State the relationship between the Jones polynomial $J(t)$ of a knot and the Jones polynomial $J_R(t)$ of its reflection. Using the Jones polynomial $J(t)$ you found in part (c), can you determine whether or not the knot in part (b) is chiral?

Question 3. (25 marks)

- (a) (5 marks) Given a curve $\gamma(t)$, define the orthonormal triple τ, ν, β and state without proof a formula for ν when $\gamma(t)$ is a unit-speed curve.
- (b) (14 marks) Consider the curve

$$\gamma(t) = \left(\frac{4}{5} \cos t, 1 - \sin t, -\frac{3}{5} \cos t \right).$$

Prove that $\gamma(t)$ is a unit-speed curve and compute τ, ν, β as well as the curvature K and the torsion T for this curve.

- (c) (6 marks) Suppose that $\gamma(t)$ is a curve such that the torsion T satisfies $T = 0$. Prove that

$$\frac{d}{dt} (\gamma(t) \cdot \beta) = 0.$$

You may use the formula for $\dot{\beta}$ given in lecture.

Question 4. (25 marks)

- (a) (3 marks) Define what it means for a curve to be regular.
- (b) (10 marks) Let $a > 0$ and $b > 0$ be constants and consider the curve

$$\gamma(t) = (a \cos t, b \sin t, 0).$$

Compute the signed curvature K_S for $\gamma(t)$ and use it to evaluate the integral

$$\int_0^{2\pi} \frac{ab}{a^2 \sin^2 t + b^2 \cos^2 t} dt.$$

- (c) (12 marks) Prove that if $\gamma(t)$ is a unit speed curve such that for some constant C it is true that $\ddot{\gamma} = C\gamma$, then γ is part of a circle.

Question 5. (25 marks)

Let $\lambda > 0$ be a constant and consider the surface patch

$$\sigma(u, v) = (v \cos u, v \sin u, \lambda u), \quad -\infty < v < \infty, 0 < u < 1.$$

- (a) (5 marks) Sketch the surface determined by σ . Determine whether or not σ is a regular surface patch.
- (b) (10 marks) Find the first and second fundamental forms of σ .
- (c) (10 marks) Find the Gaussian curvature of σ .

Question 6. (25 marks) Let $\sigma(u, v)$ be a surface patch with unit normal N_σ and first fundamental form

$$F_I = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

and let $\gamma(t) = \sigma(u(t), v(t))$ be a unit-speed curve.

(a) (9 marks) Prove that

$$N_\sigma \times \sigma_u = \frac{E\sigma_v - F\sigma_u}{\sqrt{EG - F^2}} \quad \text{and that} \quad N_\sigma \times \sigma_v = \frac{F\sigma_v - G\sigma_u}{\sqrt{EG - F^2}}.$$

(b) (3 marks) Use part (a) and the identity $\dot{\gamma} = \dot{u}\sigma_u + \dot{v}\sigma_v$ to prove that

$$N_\sigma \times \dot{\gamma} = \frac{\dot{u}(E\sigma_v - F\sigma_u) + \dot{v}(F\sigma_v - G\sigma_u)}{\sqrt{EG - F^2}}.$$

(c) (3 marks) Prove that

$$\ddot{\gamma} = \ddot{u}\sigma_u + \ddot{v}\sigma_v + \dot{u}^2\sigma_{uu} + 2\dot{u}\dot{v}\sigma_{uv} + \dot{v}^2\sigma_{vv}.$$

(d) (10 marks) Use parts (b) and (c) to prove that, if $\sigma_{uu} = \sigma_{uv} = \sigma_{vv} = 0$, then the geodesic curvature defined to be $K_g = \ddot{\gamma} \cdot (N_\sigma \times \dot{\gamma})$ satisfies

$$K_g = (\ddot{v}\dot{u} - \ddot{u}\dot{v})\sqrt{EG - F^2}.$$